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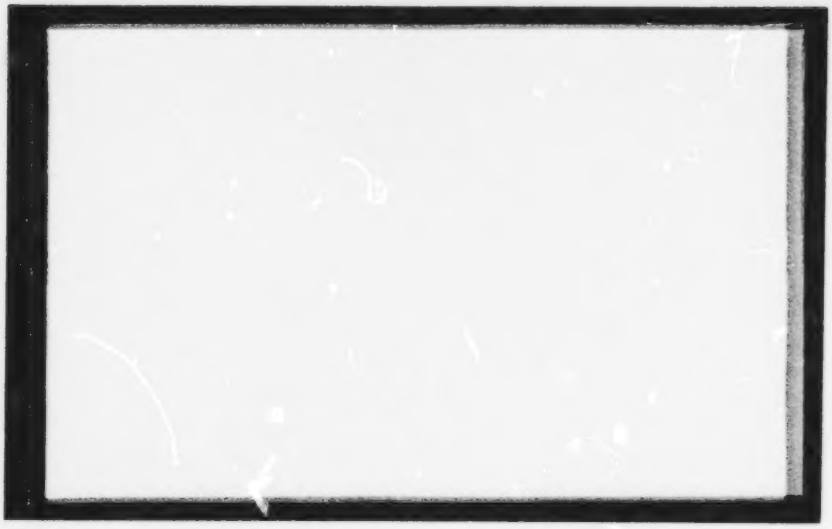

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A Unified Approach for the Solution of Network Equilibrium Problems

by

Caroline Fisk*and Sang Nguyen**

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*Civil Engineering Department, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

**Département d'informatique et de recherche opérationnelle et CRT, Université de Montréal, H3C 3J7, Canada

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Abstract

We present a general framework for formulating network equilibrium problems in which the equilibrium conditions are given by a system of non-linear equations. Existence and uniqueness results are presented, and iterative solution techniques with well-defined convergence properties are suggested. For illustrative purposes, the results are applied to a specific two mode equilibrium model.

Introduction

Over the past five years, a considerable research effort has been focused on deriving transportation planning models which couple spatial and mode travel choices with the route choice equilibrium conditions on the network. In this modeling approach the travel costs used in the choice model are those resulting from an assignment of the predicted demand on the network, giving rise to a self-consistent model which takes into account the effects of traffic congestion on travel choices. This feedback effect implies that the equilibrium conditions are given by the nonlinear relationship

$$d^* = H(u(d^*)) \tag{1}$$

where $H(u)$ represents the travel demand model as a function of the travel cost vector u and $u(d^*)$ is obtained by assigning the equilibrium demand d^* onto the network.

Most frequently, assumptions are made on the analytical form of the network link costs and the demand function, so that both the equilibrium conditions on the network, and the demand equation, can be derived as the Kuhn-Tucker conditions of a nonlinear optimization problem. In this form, the model is more amenable to analysis, and the solution can be relatively easily obtained using nonlinear programming techniques. However, an optimization formulation only exists under certain restrictive conditions which limit the potential of the model to accurately represent interactions between different traffic flows on the network. Specifically, as shown by Dafermos [1], one requirement is that the Jacobian matrix of the link cost vector be symmetric, implying that a traffic stream s_1 say, must have the same impedence potential on a stream s_2 as vice-versa.

This is not the case in reality, where priority controls are set up on a network to favor highly traveled links over lesser traveled links. Also, asymmetry arises in interactions between different types of vehicles such as automobiles and trucks and even more significantly, between automobile and transit passenger flows on the same network link.

If asymmetric interactions are incorporated into the model, it is no longer optimization based and alternative solution techniques have to be derived. The most direct way of tackling this problem is to formulate the model within a mathematical framework for which solution techniques already exist. For example, Aashtiani and Magnanti [2] have shown that the extended (i.e., asymmetric) equilibrium model can be formulated as a nonlinear complementarity problem. This has led to several important existence and uniqueness results but has not produced a practicable solution method [3]. Consequently, Aashtiani has proposed a heuristic scheme based on linearization of the costs; this method appears to work well but does not yet have proven convergence properties.

More recently, Dafermos [4] has derived similar existence and uniqueness results by showing that the extended model can be formulated as a variational inequality. A convergent solution algorithm derived from this theory is also suggested.

In related work, Abdulaal and LeBlanc [5] have proposed a solution algorithm which iteratively solves a standard (optimization based) equilibrium problem, obtained by fixing in each iteration the off-diagonal

flows on each link; these are updated for the succeding iteration using the current solution.*

Another heuristic solution algorithm for a specific two-mode (car/public transit) network equilibrium problem has been proposed by Florian [7]. Analysis of this algorithm by Fisk and Nguyen [8] showed that it was equivalent to solving a system of nonlinear equations by the method of successive approximations for which convergence properties can be established. We are going to generalize these results here and explore properties of extended network equilibrium models using the nonlinear relationship (1) as an expression of the equilibrium conditions for the model; the network equilibrium conditions are implicit in (1) since $u(d^*)$ is obtained by assigning d^* onto the network. Our model is quite general; H can incorporate mode, spatial, ans whether-or-not to travel decisions, and it may be probabilistic or direct demand based. As will be shown, the framework adopted here will lead to results on existence and uniqueness, and to the formulation of solution algorithms extracted from the theory of nonlinear equations.

The Network Equilibrium Model

We consider a transportation network consisting of a set of directed links, a set of nodes of which a subset serve as traffic origins and traffic destinations, and several distinct travel modes. L designates the number of links, N the number of O/D (origin/destination) pairs, and M the number of modes; the indices ℓ, n, and m are used respectively to designate arbitrary elements of these sets.

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*More recently, Dafermos [17] and Florian [18] have studied the convergence of this approach.

d designates an O/D travel demand vector by mode; it has elements d_{mn}, m=1, ..., M; n=1, ..., N. The set of all possible demands is closed and convex, and is denoted by $\mathcal{D} \subset R^{MN}$.

For a given $d \subset \mathcal{D}$, there is a closed, convex set $F(d) \subset R^{LM}$ of feasible link flow patterns. Any such pattern $f \subset F(d)$ will satisfy the non-negative condition

$$f \geq 0$$

and the requirement that the flow by mode be conserved at each of the network nodes.

A user cost $c_{\ell m}$ is associated with each link ℓ and mode m, and is considered to be a continuous single valued function of the link flow vector f; i.e., it can be represented as a function

$$c_{\ell m}(f) \quad \text{for all } \ell, m, \text{ and } f \in R^{LM}.$$

$c_{\ell m}(f)$ is assumed to be continuous and finite when $\|f\| < \infty$. This latter condition is not restrictive since each link cost function would be defined so that the cost increases rapidly once the effective link capacity is exceeded.

We define A(d) to be an assignment process; in other words A maps a point $d \subset \mathcal{D}$ into a point $f^*(d) \subset F(d)$, where $f^*(d)$ is an equilibrium flow pattern corresponding to the demand d and the given assignment process. The best known process is when path choice is user optimized; other possible procedures include a system optimal assignment and a multi-modal generalization of the logit assignment with congestion described in Fisk [9].

$u(d): \mathcal{D} \subset R^{MN} \rightarrow R^{MN}$ is defined to be the disutility or travel cost

vector, obtained by assigning the demand d onto the network. u is assumed to be unique and to vary continuously with d; depending on the type of process A, these requirements impose some restrictions on the classes of link cost functions that can be used and on the relationship between u and the link costs. For example, if A is the user equilibrium assignment process and u_{mn} is the cost on the shortest route(s) by mode m between n, it is sufficient that c be continuous and monotone on R_+^{LM} as can be deduced from the results of Aashtiani and Magnanti [2] and Fang [10]. If one also wants to be ensured of a unique equilibrium flow solution, the monotonicity requirement can be strengthened to strict monotonicity (see Aashtiani and Magnanti [2] and Smith [11]).

The demand function H(u) is assumed to be a continuous bounded function of u so that a closed convex set $\mathcal{D}$ can be defined such that $H(u) \subset \mathcal{D} \subset R_+^{MN}$. Since u depends on d via the assignment process A(d), and u(d) is continuous, H can be viewed as a continuous mapping of $\mathcal{D}$ onto $H\mathcal{D} \subset \mathcal{D}$. The point d* is then an equilibrium solution of the model if it is a fixed point of H; i.e., if (1) is satisfied. Consequently, the problem reduces to that of finding a fixed point of H, or in other words, of solving the nonlinear system of equations

$$d = H(u(d)) \tag{2}$$

Existence and Uniqueness Properties

In this section we show that a model solution exists and is unique if the demand function satisfies certain properties.

Theorem 1. An equilibrium solution to the system of nonlinear equations (2) exists.

Proof. H is continuous on the closed, convex set $\mathcal{D}$ and for any $d\varepsilon\mathcal{D}$, $H(d)\subset\mathcal{D}$. It then follows from Brouwer's fixed point theorem that H has a fixed point in $\mathcal{D}$.‖

Let $F(d) = 0$ be a system of equations equivalent to (2) (e.g., $F(d)=d-H(d)$), then it follows from Theorem 1 that F has at least one fixed point in $\mathcal{D}$. This point will be unique if F is one-to-one on $\mathcal{D}$, i.e., if

$$F(d^1) \neq F(d^2) \quad \text{for } \forall d^1,\ d^2\varepsilon\mathcal{D}, d^1\neq d^2.$$

A sufficient condition for this to be satisfied is that F(d) be strictly monotone on $\mathcal{D}$, i.e., that

$$[F(d^1)-F(d^2)]\ [d^1-d^2] > 0 \quad \text{for } \forall d^1,d^2\varepsilon\mathcal{D}, d^1\neq d^2.$$

The requirement is a generalization of the strict convexity requirement for optimization based models. In this case strict monotonicity follows directly from the fact that F can be obtained as the gradient mapping of a strictly convex functional.

This leads us to:

Theorem 2. A unique solution to the system (2) of nonlinear equations exists if $F(d)=d-H(d)$ is striclty monotone on $\mathcal{D}$.

To summarize our conditions for the existence and uniqueness of a solution d^* to (2) we have:

i) For any $d\varepsilon\mathcal{D}$, A(d) must produce a unique travel cost vector u which is a continuous function of d;

ii) H(u) is continuous and bounded for nay $u\varepsilon R_+^{MN}$;

iii) $F(d) = d-H(d)$ is a strictly monotone mapping on $\mathcal{D}$.

If A(d) corresponds to a user optimal assignment, our model is similar to that considered by Dafermos [4] and Aashtiani and Magnanti [2]. However, the conditions iii) given in this paper do not correspond to the monotonicity conditions for -H(u), discussed by these authors. This point will be illustrated with respect to a specific model below.

Solution methods

In proposing iterative techniques for solving (2) the following points should be born in mind:

i) for any approximation d^k of the solution d*, it is necessary to apply $A(d^k)$ in order to obtain $H(d^k)$;

ii) since H(d) is not known explicitly, and is probably costly to compute, solution methods which require derivatives of H or their approximation would not be practical;

iii) since in practice the dimensions of the problem will be extremely large, methods which require the computation of the inverse of a matrix would not be feasible.

We now suggest some general techniques which may be appropriate in this context, and in the following section examine methods which appear most suitable for some specific categories of travel demand models.

Consider the general system of nonlinear equations

$$F(x) = x - H(x) = 0 \qquad (3)$$

where $H: S \subset R^n \to R^n$ and S is some closed convex set.

The following iterative procedures* which do not require the computation of derivatives or inverses, may be adopted for solving (3):

i) Method of successive approximations. This requires solving iteratively the system:

$$x^{k+1} = H(x^k) \qquad k=0, 1, 2, \ldots, \tag{4}$$

ii) Nonlinear Gauss-Seidel. This involves solving the j^{th} equation

$$F_j(x_1^{k+1}, x_2^{k+1}, \ldots, x_{j-1}^{k+1}, x_j, x_{j+1}^{k}, \ldots, x_n^{k}) = 0$$

for x_j and setting $x_j^{k+1} = x_j$, $j=1, \ldots, n$; $k=0, 1, 2, \ldots$

iii) Nonlinear Jacobi method. In iteration k we solve the system of equations

$$F_j(x_1^{k}, \ldots, x_{j-1}^{k}, x_j, x_{j+1}^{k}, \ldots, x_n^{k}) = 0 \quad j=1, \ldots, n$$

and set $x_j^{k+1} = x_j$, $j=1, \ldots, n$; $k=0, 1, 2, \ldots$

The latter two methods can also be formulated in block form in which case instead of solving for a single variable x_j, we solve simultaneously for a block of variables. This is particularly appropriate here since problems may be conveniently decomposed by mode or origin. There is also a possible Gauss-Seidel type modification of the first method, namely to apply the process

$$x_j^{k+1} = H_j(x_1^{k+1}, \ldots, x_{j-1}^{k+1}, x_j^{k}, x_{j+1}^{k}, \ldots, x_n^{k}) \quad j=1, \ldots, n$$

or to apply this in block form. The advantage of this over (4) is that new

*All results used here relating to nonlinear equation theory can be found in Ortega and Rheinboldt [12].

information is used more rapidly and convergence could be expected to be quicker. However, this must be weighed against the difficulty in computing H(x).

All the above methods can be formulated as an iterative procedure of the form $x^{k+1} = G(x^k)$ where $G:S \subset R^n \to R^n$, and convergence to the equilibrium solution is guaranteed if G is a contraction mapping on S in some norm; i.e., if for all $x,y \in S$

$$\|G(x) - G(y)\| \leq \alpha \|x-y\| \tag{5}$$

where $\alpha < 1$. If G is differentiable on S, G is a contraction mapping if

$$\sup \|G'(x)\| < 1$$

If G is not differentiable it is sufficient that the bound of G is less than 1, i.e.,

$$\sup_{\substack{x \neq y \\ x,y \in S}} \frac{\|G(x)-G(y)\|}{\|x-y\|} \leq \alpha \leq 1 \tag{6}$$

Note that if x-H(x) is a contraction in the norm $\|\cdot\|_2$, F is strictly monotone.

Applications to travel demand models

For realistic sized network, the largest amount of computation time in algorithms of this nature, will be taken up in performing A(d), and most of this time will be spent in constructing minimum path trees. Consequently, approaches which require the least number of assignments in passing from d^k to d^{k+1} will require the least amount of computation time per iteration. On the other hand, the convergence rate is also tied to the rate at which new information is used. Given these considerations

we now consider some distinct classes of demand problems.

a) Multi-modal elastic demand equilibrium

In this problem we wish to determine the equilibrium modal split by O/D pair. The most obvious approach is to generate the sequence

$$d^{k+1} = H(d^k). \tag{7}$$

$u(d^k)$ and hence $H(d^k)$ is obtained by applying $A(d^k)$.

This method could also be applied in block form by mode. Let d_m represent the vector of O/D demands by mode m, then we could generate the sequence

$$d_m^{k+1} = H_m(d_1^{k+1}, \ldots, d_{m-1}^{k+1}, d_m^k, \ldots, d_M^k).$$

After d_m^{k+1} is obtained, an assignment of the updated demand vector would have to be performed in order to compute H_{m+1}. Since information is used more rapidly than the previous method, convergence could be expected to be faster although it would have to be approximately M times as fast to recover the extra time taken up in performing the assignments.

Neither the Gauss-Seidel or Jacobi methods appear workable for large network problems. For example if Gauss-Seidel is applied by mode, in iteration k+1 the system

$$d_m - H_m(d_1^{k+1}, \ldots, d_{m-1}^{k+1}, d_m, d_{m+1}^k, \ldots, d_M^k) = 0 \tag{8}$$

must be solved for d_m, d_m^{k+1} equated to the solution, and this repeated successively for each mode m. While the other demands are fixed in (8), path costs by other modes will change due to the interaction with path flows by mode m. Consequently the above problem is itself complex requiring an iterative procedure for solution in which an assignment of all the mode

demands would have to be performed in each iteration. The Jacobi method gives rise to similar complexity.

The complexity could be reduced by assuming that paths by other modes are not greatly affected by changes in d_m, in which case the link flows by other modes are fixed and a single mode problem is solved. This method could be expected to have satisfactory convergence properties when the interaction between flows by different modes in weak. In many cases this condition is not satisfied, e.g. in a two mode auto/public transit model, the delaying effect of auto traffic on transit passenger flows is of the same order as that of the auto traffic on itself, and transit passenger flows do not contribute significantly to delays.

b) Combined trip distribution/assignment

In this model d is the O/D travel demand vector. The most direct solution approach is to generate the sequence

$$d^{k+1} = H(d^k) \quad k = 0, 1, 2, \ldots$$

This is equivalent to applying iteratively the sequential distribution/assignment procedure, proposed prior to the development of optimization formulations for the combined distribution/assignment model, to achieve consistency between the distribution and assignment stages of a planning model. Our analysis shows that this approach fits into a rigorous mathematical framework from which analytical convergence properties can be obtained.

This method could be decomposed by origin although it would be necessary to perform an assignment after new demands for each origin are computed. However if the interactions between flows from different

origins are not significant, it would suggest that as an approximation, link flows from other origins could be fixed in the manner suggested in a).

c) Combined distribution/mode split/assignment

As with the other models, this could be solved as

$$d^{k+1} = H(d^k)$$

or decomposed by mode and/or origin and solved in block form.

In the above discussion we have assumed that the system of nonlinear equations takes the form $d = H(d)$. However in certain cases it may be more convenient to express these equations in another form before applying a solution algorithm, the most obvious alternative being $d = H^{-1}(d)$; in this acse we would use the iterative procedure $d^{k+1} = H^{-1}(d^k)$ instead of (7).

As an illustration of the approaches proposed above, we now consider a specific equilibrium model.

The logit mode split/assignment model

The model considered here couples mode choice (car/public transit) with the equilibrium conditions on the network. The basic essentials of the model are given below; further details can be found in [7] and [13].

$$d_{mn} = H_{mn}(u) = \frac{D_n \, e^{x_{mn} - \beta_m \mu_{mn}}}{\sum_m e^{x_{mn} - \beta_m \mu_{mn}}} \qquad \begin{matrix} m = 1, 2 \\ n = 1, \ldots, N \end{matrix} \qquad (9)$$

where D_n is the total demand for travel between n,

x_{mn} are flow independent explanatory variables,

β_m are calibration parameters which determine the weight of the auto and transit costs in the demand function,

μ_{mn} is the travel cost by mode m between O/D pair n.

Automobiles and transit passengers choose their routes in a user optimal fashion. Link costs depend on the auto and transit vehicle flows; the flow $\bar{f}_\ell$ on link ℓ which determines the link cost, is given by the auto flow plus a constant term taking into account transit flows, and the auto link cost function $c_{\ell 1}(\bar{f}_\ell)$ is a strictly increasing, continuously differentiable function of $\bar{f}_\ell$ with $c_{\ell 1}(\bar{f}_\ell) < \infty$ for $\bar{f}_\ell < \infty$.

The transit link cost is given by the auto cost plus a constant penalty term which takes into account stops etc.

Transit passenger flows are assumed not to affect link travel times, so that u depends only on the auto demand. To proceed further it is convenient to rewrite (9) as

$$\bar{d}_n = \bar{H}_n(\bar{d}) \qquad n = 1, \ldots, N \tag{10}$$

where

$$\bar{H}_n(\bar{d}) = \frac{D_n}{1+e^{\bar{x}_n} e^{\bar{\beta}[\bar{u}_n(d)-\bar{v}_n(\bar{d})]}} \qquad n = 1, \ldots, N \tag{11}$$

$\bar{d}$ is the auto demand vector,

$\bar{x}_n = x_{2n} - x_{1n}$,

$\bar{u}_n = u_{1n}$

$\bar{v} = \frac{\beta_2}{\beta_1} u_{2n}, \qquad n = 1, \ldots, N$

$\bar{\beta} = \beta_1$

$\bar{c}_\ell = c_{\ell 1} \qquad \ell = 1, \ldots, L$

Also $\bar{D}$ is the set $\{\bar{d}: 0 \le \bar{d}_n \le D_n,\ n=1,\ldots,N\}$.

Thus (9) has been rewritten as a single mode elastic demand problem; to facilitate notation the bar above the variables is dropped in the remainder of this section.

It was shown in [13] that the existence requirements of theorem 1 are satisfied for this problem. Also in this reference, uniqueness properties of the system of equations (10) were discussed by rewritting it in the form

$$F_n(d) = 0 \qquad n = 1,\ldots, N, \tag{12}$$

where

$$F_n(d) = u_n(d) - v_n(d) - \frac{1}{\beta} \log \frac{D_n}{D_n - d_n} + \frac{x_n}{\beta} \tag{13}$$

As an application of the solution approaches discussed in the two preceding sections, we now consider two possible algorithms, both of which are straightforward to implement. Firstly, (10) could be solved using successive approximations by generating the sequence

$$d^{k+1} = H(d^k). \tag{14}$$

The compute $H(d^k)$, a user optimal assignment of the auto demands d^k is performed, producing $u(d^k)$ and $v(d^k)$, and $H_n(d^k)$ is then calculated directly from (11). The sequence d^0, d^1, d^2, ... will converge to the equilibrium solution d^* if H is a contraction mapping on the set of feasible auto demands (or in some neighbourhood of d^*, provided d^0 lies in this neighborhood).

A second algorithm can be formulated by rewriting (12) as

$$v_n(d) = G_n(d) \stackrel{\text{def}}{=} u_n(d) - \frac{1}{\beta} \log \frac{D_n}{D_n - d_n} + \frac{x_n}{\beta} \qquad n = 1, \ldots, N \tag{15}$$

It was shown in [13] that u(d) is a monotone mapping on $\overline{\mathcal{D}}$ and hence that G(d) is strictly monotone on $\overline{\mathcal{D}}$. Consequently G is invertible and (15) is equivalent to

$$d = G^{-1}(v(d)) \tag{16}$$

If this is solved using successive approximations the sequence

$$d^{k+1} = G^{-1}(v(d^k)) \tag{17}$$

is generated. In other words the auto demands in iteration k + 1 are obtained by assuming transit costs to be fixed at the values found in interation k. Since the transit costs are constant, the auto demands can be obtained by solving an elastic demand optimization problem of the form proposed by Bechmann et al. [14]. Convergence is obtained if G^{-1} is a contraction on $G^{-1}\overline{\mathcal{D}} \subset \overline{\mathcal{D}}$ or in some neighborhood of d^* provided d^0 lies in the neighborhood.

This algorithm is identical to the heuristic method suggested by Florian [7] in the original presentation of this model. It has a considerable advantage over (14) because the solution for iteration k can be used as an initial solution to the elastic demand problem of iteration k + 1. Experience has shown that it is a good feasible solution, and in fact that the method converges rapidly [15].

The convergence requirements for these algorithms are that H and G^{-1} be contractions in some norm on $S(d^*, \delta) \subset \overline{\mathcal{D}}$ where $S(d^*, \delta)$ is the closed ball $\{d \varepsilon \overline{\mathcal{D}}: \| d-d^* \| \leq \delta\}$ and $\delta > 0$. If the initial approximation d^0 lies in S, then all iterates $d^1, d^2, \ldots$ defined by (14) and (17) will converge to d^*. For the first algorithm we have.

Theorem 3. $H: \overline{\mathcal{D}} \subset R^n \rightarrow \overline{\mathcal{D}}$ is a contraction mapping on $S(d^*, \delta)$ if

$$\beta D_n L_n \sum_{j=1}^{N} \left| \frac{\partial u_n}{\partial d_j} - \frac{\partial v_n}{\partial d_j} \right| < 1 \quad \text{for } n = 1, \ldots, N \text{ and } \forall d \in S$$

where $L_n = d_n(D_n - d_n)/ D_n^2$

Proof. In [8] it was shown that u(d) and v(d) are piecewise differentiable on $\overline{\mathcal{D}}$. Singularities exist at points where there is a change in the number of paths at equilibrium with nonzero flow, although left and right partial derivatives are defined at these points. Consequently H is piecewise differentiable on $\overline{\mathcal{D}}$; the convention is adopted that the partial derivative of variables at singular points is given by the left partial derivative. H is then a contraction mapping on S if $\|H'(d)\| < 1$ for all $d \in S$ and some arbitrary norm $\|\cdot\|$ (see [12] chapter 5).

The components of H'(d) are

$$H'_{nk} = \frac{\partial H_n}{\partial d_k} = - L_n D_n \beta \left[\frac{\partial u_n}{\partial d_k} - \frac{\partial v_n}{\partial d_k} \right]$$

Since $\| H'\|_\infty \overset{\text{def}}{=} \max_{1 \le n \le N} \sum_k |H'_{nk}|$, H is a contraction on S with respect to the norm $\|\cdot\|_\infty$ if

$$\beta L_n D_n \sum_{k=1}^{N} \left| \frac{\partial u_n}{\partial d_k} - \frac{\partial v_n}{\partial d_k} \right| < 1 \quad \text{for } n = 1, \ldots, N \text{ and } \forall d \in S. \; \|$$

This condition is similar to the condition of diagonal dominance on F, namely that

$$\left|\frac{\partial u_n}{\partial d_n} - \frac{\partial v_n}{\partial d_n} + \frac{1}{\beta L_n D_n}\right| > \sum_{j \neq n} \left|\frac{\partial u_n}{\partial d_j} - \frac{\partial v_n}{\partial d_j}\right| \quad n = 1, \ldots, N.$$

Since the partial derivative terms are of the same order, it requires that the diagonal term in F_n, $-\frac{1}{\beta} \log \frac{D_n}{D_n - d_n}$, varies at a much greater rate from changes in d than the cost terms. Some arguments to this effect based on data collected for the City of Winnipeg, are presented in [13].

Similar conditions can be derived for the second algorithm.

Theorem 4. The mapping $G^{-1}(d)$, defined in (15), is a contraction on S if

$$\beta D_n L_n \|v'\|_2 < 1 \qquad n = 1, \ldots, N \text{ and } \forall d \in S.$$

Proof. From the inverse function theorem

$$(G') = (G')^{-1} v$$

and so

$$\|(G^{-1})'\| \leq \|(G')^{-1}\| \, \|v'\| .$$

Let $\|\cdot\|_2$ designate the Euclidean norm; since G is strictly monotone, G' is positive definite. Also, as shown in [8], u' is symmetric so that G' is symmetric. Consequently, if λ_n (n=1,...,N) designate the eigenvalues of G',

$$\|G'\|_2 = \max_{1 \leq n \leq N} \lambda_n$$

and

$$\|(G')^{-1}\|_2 = \frac{1}{\min\limits_{1 \leq n \leq N} \lambda_n}$$

Thus $\|(G^{-1})\|_2 < 1$ if $\|v'\|_2 < \min_{1\le n\le N} \lambda_n$ (18)

G' is the sum of a diagonal matrix with elements $\delta_{kn}/\beta D_n L_n$, and the positive semidefinite matrix u'. It can be shown that in this case (see Lancaster [6] for example), $\lambda_n \ge \frac{1}{\beta D_n L_n}$ and thus

$$\min_{1<n<N} \lambda_n \ge \min_{1<n<N} \frac{1}{\beta D_n L_n} = \frac{1}{\max_{1\le n\le N} \beta D_n L_n}$$

Consequently (18) is satisfied if

$$\beta D_n L_n \|v'\|_2 < 1 \qquad \text{for } n = 1, \ldots, N$$

and the theorem is proved if this holds for all points $d \in S$. ‖

As with theorem 3, this requires that the diagonal term in F vary at a much greater rate than the cost term v. The conditions obtained above are sufficient (but not necessary) for the iterative algorithms to converge; further implications of the requirement that H be a contraction mapping will be studied in future research.

Discussion

In addition to the algorithms presented above, two heuristic solution approaches have been implemented on large scale networks and found to function satisfactorily, namely the methods proposed by Abdulaal and LeBlanc [5], and Aashtiani and Magnanti [3]. However analytical convergence properties for these methods have not been derived . On the other hand, the method based on the variational inequality formulation (Dafermos [4]),

does have well-defined convergence properties, but the algorithm has not been tested on realistic size problems, and also cannot be applied to models where the demand functions H(u) are not invertible.

For the model discussed in the previous section, the total link cost vector, c(f) is a monotone function on F (this result follows from the fact that transit passenger flows do not affect the travel costs), and -H(u) is monotone on $\mathcal{D}$. Theorem 6.2 in [2] requires either c or -H to be strictly monotone to produce uniqueness results.

In summary, the approach considered here allows us to obtain uniqueness results for models which cannot be analysed within the frameworks of the nonlinear complementarity problem and variational inequalities. It results in solution algorithms which are straightforward to apply and have been shown to work well in one large scale application [15]. Further research is required into the implications of the contraction mapping condition for convergence; this is presently under investigation by the authors.

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